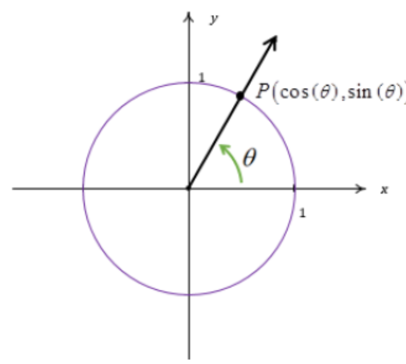
# Determining Cosine and Sine Values from the Unit Circle

Consider the **Unit Circle**, $x^2 + y^2 = 1$, as shown, with the angle $\theta$ in standard position and the corresponding arc measuring $s$ units in length. By the definition established in **Section 1.2**, and the fact that the Unit Circle has radius 1, the radian measure of $\theta$ is $\frac{s}{r} = \frac{s}{1} = S$ so we have $\theta = S$.
In order to identify real numbers with oriented angles, we make good use of this fact by essentially 'wrapping' the real number line around the Unit Circle and associating to each real number $t$, an *oriented* arc on the Unit Circle with initial point $(1, 0)$.

**Examples:** Sketch the oriented arc on the Unit Circle corresponding to each of the following real numbers.

$t = \frac{3\pi}{4}$ $\quad$ $t = -2\pi$ $\quad$ $t = -2$ $\quad$ $t = 117$

## The Cosine and Sine as Circular Functions




Consider an angle $\theta$ in standard position and let $P$ denote the point where the terminal side of $\theta$ intersects the Unit Circle. By associating the point $P$ with the angle $\theta$, we are assigning a *position* on the Unit Circle to the angle $\theta$. The $x$-coordinate of $P$ is called the **cosine** of $\theta$, written $\cos(\theta)$, while the $y$-coordinate of $P$ is called the **sine** of $\theta$, written $\sin(\theta)$.

**Examples:** Find the sine and cosine for each angle.

$\theta = 270^\circ$ $\quad$ $\theta = -\pi$ $\quad$ $\theta = 45^\circ$ $\quad$ $\theta = \frac{\pi}{6}$ $\quad$ $\theta = 60^\circ$

Knowing which quadrant an angle $\theta$ terminates in will help us determine whether $\cos(\theta)$ and $\sin(\theta)$ are positive or negative (remember $r$ is always positive).

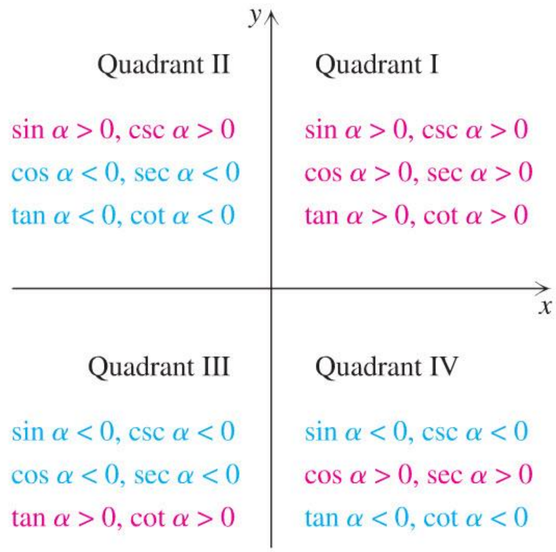
A good mnemonic to remember which functions are positive in each quadrant is "**All S**tudents **T**ake **C**alculus":

Quadrant I: All of them are positive

Quadrant II: sin and csc are positive

Quadrant III: tan and cot are positive

Quadrant IV: cos and sec are positive




**The Pythagorean Identity**

In the previous section we used the fact that $P(x, y) = (\cos(\theta), \sin(\theta))$ lies on the Unit Circle, $x^2 + y^2 = 1$. If we substitute $x = \cos(\theta)$ and $y = \sin(\theta)$ into $x^2 + y^2 = 1$, we get the identity $(\cos(\theta))^2 + (\sin(\theta))^2 = 1$. This is the Pythagorean Identity: For any angle $\theta$, $\cos^2(\theta) + \sin^2(\theta) = 1$.

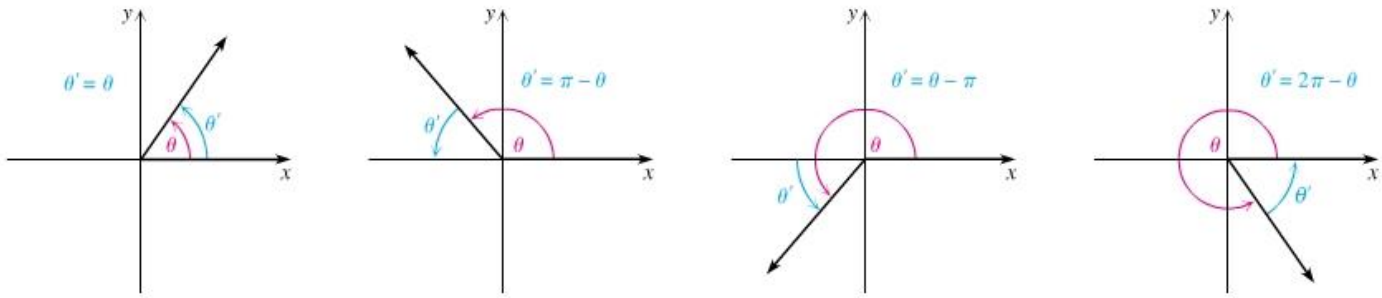
If we know the sine or cosine of an angle, then we can use the *Pythagorean Identity* to find the value of the other function of the angle. (Note: you can also figure this out by drawing a triangle and using the Pythagorean Theorem).

**Example:** Find $\sin\theta$ given that $\cos\theta = 1/4$ and $\theta$ is in Quadrant I.

**Example:** Find $\cos\theta$ given that $\sin\theta = -\sqrt{5}/3$ and $\theta$ is in Quadrant III.

**Reference Angles:** When you look at the unit circle, notice that there is a pattern to the coordinates. If you look at all the angles that are 30° away from the *x*-axis (30°, 150°, 210°, 330°), the *x*-coordinate (cosine) is $\pm\sqrt{3}/2$ and the *y*-coordinate (sine) is $\pm 1/2$.

**Definition: Reference Angle:** If $\theta$ is a nonquadrantal angle (not on an axis) in standard position, then the reference angle $\theta'$ (read "theta prime") formed by the terminal side of $\theta$ and the positive or negative *x*-axis.




**Examples:** For each given angle $\theta$, sketch the reference angle $\theta'$ and give the measure of $\theta'$ in both radians and degrees.

$\theta = 120°$ $\qquad$ $\theta = 7\pi/6$ $\qquad$ $\theta = 690°$ $\qquad$ $\theta = -7\pi/4$

**Reference Angle Theorem:** For an angle $\theta$ in standard position that is not a quadrantal angle:

$$\sin\theta = \pm\sin\theta', \qquad \cos\theta = \pm\cos\theta', \qquad \tan\theta = \pm\tan\theta',$$

$$\csc\theta = \pm\csc\theta', \qquad \sec\theta = \pm\sec\theta', \qquad \cot\theta = \pm\cot\theta'$$

where $\theta'$ is the reference angle for $\theta$ and the sign is determined by the quadrant in which terminal side of $\theta$ lies.

***Finish filling out the Unit Circle** (if not already complete)

**Examples:** Find the sine and cosine for each angle using reference angles.

$\theta = 225^{\circ}$ $\qquad$ $\theta = \dfrac{11\pi}{6}$ $\qquad$ $\theta = -\dfrac{5\pi}{4}$ $\qquad$ $\theta = \dfrac{7\pi}{3}$